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*Two-Dimensional Signal Processing With
Application to Image Restoration*

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Preface

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Abstract

A recursive technique for modeling and estimating a two-dimensional signal contaminated by noise is presented. A two-dimensional signal is assumed to be an undistorted picture, where the noise introduces the distortion. Both the signal and the noise are assumed to be wide-sense stationary processes with known statistics. Thus, to estimate the two-dimensional signal is to enhance the picture.

The picture representing the two-dimensional signal is converted to one dimension by scanning the image horizontally one line at a time. The scanner output becomes a nonstationary random process due to the periodic nature of the scanner operation. Procedures to obtain a dynamical model corresponding to the autocorrelation function of the scanner output are derived. Utilizing the model, a discrete Kalman estimator is designed to enhance the image.

Two-Dimensional Signal Processing With Application to Image Restoration

I. Introduction

In theory, image enhancement utilizing classical filtering techniques does not seem to be difficult, since the processing of signals in one dimension can usually be extended to a two-dimensional case (using the concept of linear system theory). However, the application is cumbersome and often becomes impractical when a large amount of data is to be processed. Thus, it seems desirable in many situations that a recursive technique of image enhancement be developed.

In this report, we shall consider those images that are distorted by random noise and are best characterized by statistical procedures, such as specifying their first two statistical moments associated with the random process representing the brightness level. Thus, in this case, the image enhancement becomes a problem of statistical estimation and filtering, and to enhance a distorted image is to estimate the image. The input to the estimator is the output of a horizontal line scanner scanning the image with uniform speed. The procedure is first to develop a dynamical model whose response characteristic matches

that of the scanner output in the statistical sense. The image is two-dimensional, while the scanner output is one-dimensional; thus, the model must exhibit the vertical correlation of the image. This correlation will be revealed by the oscillatory nature of the model responses.

It is assumed that the image is characterized by a stationary, two-dimensional autocorrelation function. However, the scanner output, denoted by $s(t)$, is nonstationary and nonseparable (Ref. 1) because of the scanner's periodic movement. Consequently, no finite dimensional linear dynamic model representing $s(t)$ exists. To remedy the nonseparability of the nonstationary process, $s(t)$, which is a very undesirable situation (Ref. 1), we shall generate a stochastic process whose autocorrelation function is stationary and which approximates the autocorrelation function of $s(t)$.

Since we shall be dealing with the question of realization of autocorrelation functions and thus spectral factorization, a brief background of spectral factorization is presented.

A. Spectral Factorization

The concept of spectral factorization has become increasingly more important since Wiener's original work (Ref. 2) on the subject. Basically, spectral factorization determines the equations that describe a linear system when the system is driven by white noise and the covariance of the output is known. Whenever the covariance function of a process is driven by white noise via a system of differential equations of first order, we refer to this system as a dynamical model. More specifically, given a covariance function $R(t, \tau)$, where $t \leq t_1$ and $\tau \leq \tau_1$ for some fixed t_1 and τ_1 , the factorization problem is to determine a realizable linear filter (differential equation model) that, when driven by white noise, yields $R(t, \tau)$ as its output covariance.

It is well known (Ref. 1) that, in general, no such realization may exist. However, if its existence were guaranteed, the representation (in some sense) would be unique (Ref. 3). In its most popular form, the spectral factorization would be confined to stationary situations. Then the corresponding dynamical model under consideration would be time-invariant, and the white noise forcing function must have started infinitely in the past (Ref. 3). This dynamical model would be asymptotically stable (Ref. 3). It is also desirable to deal with finite-dimensional dynamical models, implying that each linear model must possess a rational bilateral Laplace Transform. We can summarize the above discussion by the statement of Theorem 1, which we shall not prove (Ref. 4).

Theorem 1

A necessary and a sufficient condition that a stationary process $y(t)$ be representable as the output of an asymptotically stable, time-invariant, finite, dimensional linear model is that its spectral density $R(s)$ be a rational function of the form $H(s)H(-s)$, with

$$H(s) = \frac{M(s)}{\rho(s)} \quad (1)$$

for some polynomial

$$\rho(s) = s^n + \sum_{i=0}^{n-1} \alpha_i s^i$$

with all roots in the left half part of the s -plane and

$$M(s) = \sum_{i=0}^{n-1} \beta_i s^i,$$

with degree less than or equal to $n - 1$ and all roots in the left half of the s -plane, where α_i and β_i are the real coefficients. That is, $H(s)$ has all of its poles and zeros in the left half of the s -plane.

B. Determination of the Output Covariance From a Linear Dynamical Model

Consider the following dynamical model, given by

$$\begin{aligned} \dot{x} &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) \end{aligned} \quad (2)$$

where $x(t)$ is an $n \times 1$ vector, u is an $m \times 1$ vector, y is a scalar, A , B , and C are matrices of appropriate dimensions (not necessarily time-invariant), and $u(t)$ is a zero-mean white noise vector, such that

$$Eu(t)u'(\tau) = K\delta(t - \tau) \quad (3)$$

where K is an $m \times m$ symmetrical matrix and prime denotes the transpose.

It is desired to calculate the output covariance (an autocorrelation, since $y(t)$ is of zero mean) $Ey(t)y(\tau)$, given by

$$Ey(t)y(\tau) = C(t)Ex(t)x'(\tau)C'(\tau) \quad (4)$$

Let the random variable $x(t_0)$, where t_0 is the initial time, be statistically independent of $u(t)$. It is well known (Ref. 5) that the solution of $x(t)$ is given by

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^t \Phi(t, \tau)B(\tau)u(\tau)d\tau \quad (5)$$

where $\Phi(t, \tau)$ is the state transition matrix; i.e.,

$$\frac{\partial \Phi(t, \tau)}{\partial t} = A(t)\Phi(t, \tau) \quad (6)$$

$$\Phi(t, t) = I \quad (7)$$

Substituting $x(t)$ from Eq. (5) into (4) and performing some mathematical operations, we obtain (Ref. 3)

$$\begin{aligned} Ey(t)y(\tau) &= C(t)\Phi(t, \tau)P_x(\tau)C'(\tau)1(t - \tau) \\ &\quad + C(t)P_x(t)\Phi'(t, \tau)C'(\tau)1(\tau - t) \end{aligned} \quad (8)$$

$$P_x(t) = Ex(t)x'(t) \quad (9)$$

where $1(t)$ is a unit step function.

From the dynamical model (Eq. 2), $P_x(t)$ can be shown to be the solution of the differential equation (Ref. 3)

$$\dot{P}_x = AP_x + P_xA' + BKB' \quad (10)$$

where the covariance $P_x(t_0)$ must be given.

C. Independence of Estimation Problem of a Particular Coordinate System

In spectral realization, $y(t)$, given by Eq. (2), is the signal without any noise contamination. Often, we receive a contaminated observation $z(t)$, given by

$$z(t) = y(t) + n(t) \quad (11)$$

where $n(t)$ is additive noise which is assumed to be uncorrelated with $y(t)$. Kailath (Ref. 1) and Anderson (Ref. 6) show that the only information necessary for recursive estimation is the knowledge of $Ey(t)y(t+\tau)$ and $Ez(t)z(t+\tau)$. That is, the solution of recursive estimation in the mean-square sense is independent of the particular coordinate system for model $z(\cdot)$ and $y(\cdot)$ processes; hence, a unique solution associated with minimum mean-square estimation can be obtained where the models for the processes are not given in advance. All these models are related to one another by a linear transformation. For example, if

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) \\ y &= Cx(t) + v(t) \end{aligned} \quad (12)$$

and

$$\begin{aligned} x^* &= A^*x^*(t) + B^*u^*(t) \\ y &= C^*x^*(t) + v(t) \end{aligned} \quad (13)$$

correspond to the same realization, then there exists a linear transformation $T(t)$ such that

$$x^*(t) = T(t)x(t) \quad (14)$$

and

$$\hat{x}^* = T(t)\hat{x}(t) \quad (15)$$

where \hat{x} and \hat{x}^* are the estimates corresponding to Eqs. (12) and (13), respectively.

The covariance estimates can be obtained accordingly.

II. Recursive Image Estimation

A. Procedure Outline

The enhancement of images that are characterized only by statistical data where the picture contains additive noise is considered in this section. The random process representing the output scanner is characterized by the output of a dynamical model with white noise input. The dynamical model describes the first-order vector Markov process. The procedure of Kalman filtering is then utilized to recursively determine the minimum mean-square error estimate of the image. The result is also extended to obtain the smoothing of data. Two examples (Ref. 7), one with very high SNR, are used to illustrate the effectiveness of the procedure. In what follows, the image is assumed to be a two-dimensional, stationary correlation function of zero mean. Thus, the autocorrelation function and the covariance become identical. The statistical information about the image and the noise is assumed to be known and uncorrelated, and the noise is additive.

B. Derivation of Autocorrelation Function of Scanner Output

Let us scan a picture horizontally using an optical scanner denoted by $s(t)$. Let the horizontal position (a continuous variable) be denoted by z , where $0 \leq z \leq Z$, and the vertical variable by an integer $n = 1, 2, \dots, N$ representing the n th scanned line. The brightness function is defined by $b(z, n)$. Let us assume, without any loss of generality that $b(z, n)$ is of zero mean. The random process $b(z, n)$ is assumed to be wide-sense stationary, with the autocorrelation function defined by

$$Eb(z_2, n_2)b(z_1, n_1) \doteq R(z_2 - z_1, n_2 - n_1) \doteq R(z, n) \quad (16)$$

Assume that the scanner output $s(t)$ has a horizontal speed $v = 1$ and, without any loss of generality, that the vertical movement takes zero time. Let us determine $Es(t)s(t+\tau)$ in terms of $R(z, n)$ and Z . The variables t and τ can be equivalently expressed by

$$\begin{cases} t = jT + \sigma, & j = 0, 1, 2, \dots, N-1, & 0 \leq \sigma \leq T \\ \tau = iT + \gamma, & i = \dots, -1, 0, 1, \dots \\ 0 \leq t + \tau \leq NT, & & 0 \leq \gamma \leq T \end{cases} \quad (17)$$

where $T = Z$ is the time required to traverse one horizontal line. The scanner output can now be written as

$$s(t) = b(\sigma, j+1), \quad s(t+\tau) = \begin{cases} b(\sigma + \gamma, i + j + 1) & \text{if } \sigma + \gamma \leq T \\ b(\sigma + \gamma - T, i + j + 2) & \text{if } \sigma + \gamma > T \end{cases} \quad (18)$$

Now, utilizing Eqs. (16) and (17), we can obtain

$$Es(t)s(t+\tau) = \begin{cases} R(\gamma, i) & \text{if } \sigma + \gamma \leq T \\ R(\sigma + \gamma - T, i + j + 2) & \text{if } \sigma + \gamma > T \end{cases} \quad (19)$$

It is clear that $Es(t)s(t+\tau)$ is a function of both σ and γ , or equivalently, of t and τ ; thus, it must be nonstationary. The nonstationarity is due to the edge condition. A simple check shows that $Es(t)s(t+\tau)$ is also periodic and a nonseparable function. It can be demonstrated that no finite-dimensional linear realization of this nonseparable autocorrelation exists (Ref. 3).

We shall now seek to generate a random process denoted as $q(t)$ such that it has a stationary autocorrelation function which approximates the autocorrelation of the process $s(t)$. To generate $q(t)$, we proceed as follows. For a given t , $q(t)$ is defined by

$$q(t) = s(jT + \xi) \quad (20)$$

where ξ is assumed to be uniformly distributed over $[0, T]$. Now we shall prove the following theorem (Ref. 8).

Theorem 2

The random process $q(t)$ defined by Eq. (20) is stationary.

Proof

It is easy to verify that

$$Eq(t) = 0$$

by the construction of $q(t)$.

Next, we must prove that $Eq(t)q(t+\tau)$ is a function of τ (or equivalently, γ) only. To accomplish this end, we calculate the correlation function of the process $q(t)$:

$$\begin{aligned} Eq(t)q(t+\tau) &= E_{\xi}E_{\sigma}[s(jT + \xi)s(jT + \xi + iT + \gamma)] \\ &= \frac{1}{T} \int_0^T E_{\sigma}[s(jT + \xi) \\ &\quad \times s(jT + \sigma + iT + \gamma)] d\sigma \end{aligned} \quad (21)$$

This equation is obtained by utilizing Eq. (20) and $\tau = iT + \gamma$, which is given by (17), and the fact that ξ is uniformly distributed over $[0, T]$. The subscripts s and ξ in (21) denote the expectation with respect to s and ξ , respectively. From Eqs. (19) and (21), one obtains

$$\begin{aligned} Eq(t)q(t+\tau) &= \frac{1}{T} \left[\int_0^{T-\gamma} R(\gamma, i) d\xi \right. \\ &\quad \left. + \int_{T-\gamma}^T R(T-\gamma, i+1) d\xi \right] \\ &= \frac{T-\gamma}{T} R(\gamma, i) + \frac{\gamma}{T} R(T-\gamma, i+1) = r(\tau) \end{aligned} \quad (22)$$

where $Eq(t)q(t+\tau)$ is defined as $r(\tau)$, which is a function of τ (or γ) only.

It is interesting to note that the correlation function of $q(t)$, namely $r(\tau)$, can also be obtained by averaging the autocorrelation function of $s(t)$ over one period. However, it is important to mention that such averaging over the subintervals of a period may not give rise to a stationary autocorrelation function, and furthermore, it may not yield an autocorrelation function at all.

As an example, consider a scalar random process characterized by a scalar differential equation

$$\begin{aligned} \dot{x} &= -x + u \\ y(t) &= \cos(t)x(t) \end{aligned}$$

where the initial state $x(0) = 1/2$ and

$$\begin{aligned} Eu(t) &= 0 \\ Eu(t_1)u(t_2) &= \delta(t_2 - t_1) \end{aligned}$$

Then, the autocorrelation of $x(t)$ can be obtained as follows:

$$Ex(t)x(t+\tau) = \frac{1}{2} \exp(-|\tau|)$$

Thus, $Ey(t)y(t+\tau)$ is given by

$$Ey(t)y(t+\tau) = \frac{1}{2} \cos(t) \cos(t+\tau) \exp(-|\tau|)$$

which is clearly nonstationary, since the correlation function of $y(t)$ depends on both t and $t+\tau$ and is periodic (of periodicity 2π). However, if we averaged this autocorrelation over $[0, \pi/4]$, the resulting average would depend on both t and $t+\tau$.

The randomization of ξ over the period T has the intuitive appeal that all points of the picture are weighted equally.

The following salient properties of $r(\tau)$ will be used in what follows:

$$r(iT) = R(0, i) \quad (23)$$

Since $R(z, n)$ is an autocorrelation function,

$$R(0, n) \geq R(z, n) \quad (24)$$

Thus, from (22) and (23),

$$\frac{r(iT + \gamma)}{r(iT)} \leq 1; \quad \text{for all } i, \gamma \quad (25)$$

The above properties indicate that, in general, the correlation function $r(\tau)$ has a periodic nature.

Example 1

Consider a square picture subdivided into a 32×32 grid. Let $T = 1$ second and $v = 1$. The signal is a 12×12 square starting at the 13th row and 13th column. Let m and n represent specific rows and columns, respectively. The above signal is represented by the brightness level $b(m, n) = 6.1$ where the signal exists and -1 otherwise,

resulting in a zero mean sample function. As a first approximation, let us choose

$$R(x, i) = \alpha \exp(-\mu_h |z| - \mu_v |i|)$$

where α , μ_h , and μ_v are to be determined. Computation of the sample power results in $\alpha = R(0, 0) \approx 6.1$. The correlation between two adjacent grid points is calculated as 5.33, which is a value for $R(1/32, 0)$ or $R(0, 1)$. Hence,

$$R(x, i) = 6.1 \exp(-4.35|z| - 0.136|i|)$$

The correlation function is obtained by substituting the above into Eq. (22), and the plot is shown in Fig. 1.

C. Dynamical Modeling of Image Statistics

In this section, we wish to derive a differential equation model whose solution has an autocorrelation function approximating $r(\tau)$ given by Eq. (22). Since we subsequently intend to utilize a Kalman estimator, we seek a dynamical model of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \quad (26)$$

where $x(t)$ is an n -dimensional vector, $u(t)$ is a white noise vector, and $y(t)$ is the scalar signal whose autocorrelation function is $r(\tau)$.

The procedure followed is to represent an approximation to $r(\tau)$, denoted by $r_a(\tau)$, as a sum of terms such that each term can be easily modeled, since, in general, $r(\tau)$ may not have a rational bilateral transform. The properties of $r(\tau)$ may be utilized to decompose $r(\tau)$ into the product of two functions $h(\tau)$ and $r(\tau)/h(\tau)$:

$$r(\tau) = \frac{r(\tau)}{h(\tau)} h(\tau) \quad (27)$$

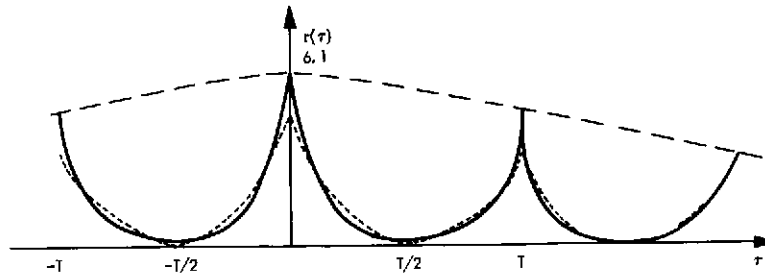


Fig. 1. Plot of $r(\tau)$ and $r_a(\tau)$ (dashed curve) as a function of τ

where $h(\tau)$ is chosen to satisfy

$$h(iT) = R(0, i); \quad \text{for all } i \quad (28)$$

Since, in many practical cases, the two-dimensional correlation function $R(z, i)$ is a monotonically decreasing function of i , a natural candidate for $h(\tau)$ is, in those instances, a combination of negative exponentials; i.e.,

$$h(\tau) = \sum_{i=1}^J l_i \exp(-\lambda_i |\tau|) \quad (29)$$

The function $p(\tau)$ is then chosen to be a periodic function approximating $r(\tau)/h(\tau)$. The approximate correlation function is

$$r_a(\tau) = h(\tau) p(\tau) \quad (30)$$

Utilizing Eqs. (23) and (28), it can be seen that the function $r(\tau)/h(\tau)$ is unity at iT and less than unity for all other τ ; furthermore, from (22) and (29), it is an even function. Hence, $p(\tau)$ is chosen to be an even function with period T . Thus, a natural candidate for this function is

$$p(\tau) = \sum_{j=0}^J a_j \cos \frac{2\pi j}{T} \tau \quad (31)$$

Consequently, an element of the function $r_a(\tau)$ has the form

$$l_i a_i \exp(-\lambda_i |\tau|) \cos \frac{2\pi j}{T} \tau \quad (32)$$

and there are $(J+1)I$ such elements.

A differential equation model with white noise input can be simply constructed (Ref. 8) to model each of these terms. Each will be a second-order system except for those corresponding to $j=0$; i.e.,

$$l_i a_0 \exp(-\lambda_i |\tau|)$$

which will be of first order. If the white noise forcing functions (one being necessary for each i, j pair) are chosen to be mutually independent, the collection of all these differential equations defines the parameters A, B, C and represents the desired model for $r_a(\tau)$.

In the course of selecting the approximate function $r_a(\tau)$, we must choose the coefficients properly, such that $r_a(\tau)$ is a correlation function. We shall either guarantee that $r_a(\cdot)$ is a positive definite function or equivalently, that the spectral density of $r_a(\tau)$ is positive (Ref. 9).

Example 2

Using example 1, let us derive a dynamic model for $r(\tau)$. Assume that the desired model has the form given by Eq. (26), and further, that

$$Eu(t)u(t+\tau)' = K\delta(\tau) \quad (33)$$

where $\delta(\tau)$ is the dirac delta function, the prime denotes the transpose, K is a positive definite matrix, and

$$Ey(t)y(t+\tau) = r_a(\tau) \quad (34)$$

Because of the exponential nature of $R(z, i)$, we choose

$$h(\tau) = R(0, 0) \exp(-0.136|\tau|) \quad (35)$$

and

$$p(\tau) = \sum_{j=0}^J a_j \cos 2\pi j \tau \quad (36)$$

In this example, we use the notation μ_v instead of 0.136.

The modeling procedure can be broken down as follows. The first term $r_a(\tau)$, namely,

$$a_0 \exp(-\mu_v |\tau|)$$

has the bilateral transform

$$\frac{2\mu_v a_0}{(s + \mu_v)(s - \mu_v)} \doteq R_1(s) \quad (37)$$

The function $R_1(s)$ can now be factored into two functions, $H_1(s)$ and $H_1(-s)$, where

$$R_1(s) = \frac{\sqrt{2a_0\mu_v}}{(s + \mu_v)} \frac{\sqrt{2a_0\mu_v}}{(s - \mu_v)}$$

and

$$H_1(s) = \frac{\sqrt{2a_0\mu_v}}{s + \mu_v}$$

Utilizing the method of Section II, a dynamic realization of $H_1(s)$ is obtained as (Ref. 10)

$$\begin{aligned} \dot{x}_1 &= -\mu_v x_1(t) + \sqrt{2a_0\mu_v} u_1(t) \\ y_1(t) &= x_1(t) \end{aligned} \quad (38)$$

The second term of $r_a(\tau)$, namely,

$$a_1 \exp(-\mu_v|\tau|) \cos 2\pi\tau$$

has the following bilateral transform:

$$R_2(s) = \frac{2a_1\mu_v[-s^2 + (2\pi)^2 + \mu_v^2]}{[(s + \mu_v)^2 + (2\pi)^2][(-s + \mu_v)^2 + (2\pi)^2]}$$

The function $R_2(s)$ can be factored out into two functions, $H_2(s)$ and $H_2(-s)$:

$$\begin{aligned} R_2(s) &= \frac{\sqrt{2a_1\mu_v}[s + \sqrt{(2\pi)^2 + \mu_v^2}]}{(s + \mu_v)^2 + (2\pi)^2} \\ &\times \frac{\sqrt{2a_1\mu_v}[-s + \sqrt{(2\pi)^2 + \mu_v^2}]}{(-s + \mu_v)^2 + (2\pi)^2} \end{aligned}$$

where $H_2(s)$ is given by

$$H_2(s) = \frac{\sqrt{2a_1\mu_v}[s + \sqrt{(2\pi)^2 + \mu_v^2}]}{(s + \mu_v)^2 + (2\pi)^2}$$

The corresponding dynamic realization of $H_2(s)$ is given as

$$\begin{aligned} \dot{x}^{(2)} &= A^{(2)}x^{(2)}(t) + B^{(2)}u^{(2)}(t) \\ y^{(2)}(t) &= C^{(2)}x^{(2)}(t) \end{aligned}$$

where the superscript denotes the model corresponding to the appropriate term. The coefficients $A^{(2)}$, $B^{(2)}$, and $C^{(2)}$ are given as

$$\begin{aligned} A^{(2)} &= \begin{bmatrix} 0 & 1 \\ -(2\pi)^2 + \mu_v^2 & -2\mu_v \end{bmatrix} \\ B^{(2)} &= \begin{bmatrix} \sqrt{2a_1\mu_v} \\ \sqrt{2a_1\mu_v}[\sqrt{(2\pi)^2 + \mu_v^2} - 2\mu_v] \end{bmatrix} \\ C^{(2)} &= [1 \quad 0] \end{aligned}$$

In general, the $(K+1)$ term of $r_a(\tau)$ is $a_k \exp(-\mu_v|\tau|) \cos 2\pi k\tau$ has the bilateral transform $R_{k+1}(s)$, given by

$$R_{k+1}(s) = \frac{2a_k\mu_v[-s^2 + (2k\pi)^2 + \mu_v^2]}{[(s + \mu_v)^2 + (2k\pi)^2][(-s + \mu_v)^2 + (2k\pi)^2]} \quad (39)$$

As before, the function $R_{k+1}(s)$ can be factored into two functions, $H_{k+1}(s)$ and $H_{k+1}(-s)$:

$$\begin{aligned} R_{k+1}(s) &= \frac{\sqrt{2a_k\mu_v}[s + \sqrt{(2k\pi)^2 + \mu_v^2}]}{(s + \mu_v)^2 + (2k\pi)^2} \\ &\times \frac{\sqrt{2a_k\mu_v}[-s + \sqrt{(2k\pi)^2 + \mu_v^2}]}{(-s + \mu_v)^2 + (2k\pi)^2} \end{aligned}$$

where

$$H_{k+1}(s) = \frac{\sqrt{2a_k\mu_v}[s + \sqrt{(2k\pi)^2 + \mu_v^2}]}{(s + \mu_v)^2 + (2k\pi)^2}$$

and the corresponding dynamical model is

$$\begin{aligned} \dot{x}^{(k+1)} &= A^{(k+1)}x^{(k+1)}(t) + B^{(k+1)}u^{(k+1)}(t) \\ y^{(k+1)}(t) &= C^{(k+1)}x^{(k+1)}(t) \end{aligned} \quad (40)$$

where

$$A^{(k+1)} = \begin{bmatrix} 0 & 1 \\ -(2k\pi)^2 + \mu_v^2 & -2\mu_v \end{bmatrix} \quad (41)$$

$$B^{(k+1)} = \begin{bmatrix} \sqrt{2a_k\mu_v} \\ \sqrt{2a_k\mu_v}[\sqrt{(2k\pi)^2 + \mu_v^2} - 2\mu_v] \end{bmatrix} \quad (42)$$

$$C^{(k+1)} = [1 \quad 0] \quad (43)$$

It can be seen that the first term of $r_a(\tau)$ is modeled by Eq. (38), which is a first-order system, and the subsequent terms by (39), which is the second-order system. Thus, to model the $(J+1)$ terms of $r_a(\tau)$, we need a

$(2J+1)$ -order system. For example, suppose the function $r_a(\tau)$ has $(J+1)$ terms, then we can incorporate the first- and second-order systems into a new system, whose parameters A , B , and C are obtained as follows:

$$A = \begin{bmatrix} -\mu_v & 0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & -[(2\pi)^2 + \mu_v^2] & -2\mu_v & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -[(2\pi(J+1))^2 + \mu_v^2] & -2\mu_v \end{bmatrix} \quad (44)$$

$$B = \begin{bmatrix} \sqrt{2a_0\mu_v} & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \sqrt{2a_1\mu_v} & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \sqrt{2a_1\mu_v} [\sqrt{(2\pi)^2 + \mu_v^2} - 2\mu_v] & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{2a_J\mu_v} \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \sqrt{2a_J\mu_v} [\sqrt{(2\pi J)^2 + \mu_v^2} - 2\mu_v] \end{bmatrix} \quad (45)$$

$$C = [1 \quad 1 \quad 0 \quad 1 \quad 0 \quad \cdots \quad 1 \quad 0] \quad (46)$$

Example 3

If in example 2 only three terms of $r_a(\tau)$ are retained, i.e., $J=2$, the resultant $r_a(\tau)$ can be written as

$$r_a(\tau) = 6.1 \exp -0.136(\tau) \sum_{j=0}^2 a_j \cos 2\pi\tau$$

If we use the Fourier series for $p(\tau)$, then a_0 , a_1 , and a_2 will be given as

$$a_0 = 0.333; \quad a_1 = 0.405; \quad a_2 = 0.101$$

A plot of $r_a(\tau)$ is shown in Fig. 1. The correlation term

$$6.1 a_0 \exp(-0.136|\tau|)$$

is modeled by

$$\dot{x}_1 = -0.136 x_1(t) + 0.732 u_1$$

The second term in the correlation is modeled by

$$\dot{x}_2 = x_3 + 0.82 u_2$$

$$\dot{x}_3 = -39.4 x_2 - 0.27 x_3 + 4.92 u_2$$

and the third term is modeled in a similar manner. The terms u_1 , u_2 , and u_3 represent independent white-noise terms, each with zero mean and correlation function $\delta(\tau)$, where δ is the dirac delta function. The final results are:

$$A = \begin{bmatrix} -0.136 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -39.4 & -0.27 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -157.7 & -0.27 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.743 & 0 & 0 \\ 0 & 0.820 & 0 \\ 0 & 4.92 & 0 \\ 0 & 0 & 0.410 \\ 0 & 0 & 5.04 \end{bmatrix}$$

$$C = [1 \quad 1 \quad 0 \quad 1 \quad 0]$$

Often, two-dimensional stationary correlation functions can be approximated by a combination of two-dimensional stationary correlation functions of the form

$$R(x, i) = R(0, 0) \exp(-\mu_h |x| - \mu_v |i|) \quad (47)$$

Because of the importance of $R(x, i)$ as given by Eq. (47), we shall discuss this special autocorrelation function below.

Calculating $r(\tau)$ (given by Eq. 22), one obtains

$$r(\tau) = \frac{T - \gamma}{T} \exp(-\mu_h |\gamma| - \mu_v |i|) + \frac{\gamma}{T} \exp(-\mu_h |T - \gamma| - \mu_v |i + 1|) \quad (48)$$

where

$$\tau = iT + \gamma, \quad 0 \leq \gamma \leq T$$

Now, let us define a risk function $\mathcal{R}(\cdot)$ such that

$$\mathcal{R}(r) = \int_0^{NT} [r(\tau) - r_a(\tau)]^2 d\tau \quad (49)$$

and

$$r_a(\tau) = \sum_{j=0}^J a_j \exp(-\mu_v |\tau|) \cos \frac{2\pi j}{T} \tau \quad (50)$$

We can select the coefficients a_j such that the risk function $\mathcal{R}(r)$ is minimized. For simplicity, we shall assume that $T = 1$. It can be shown that $\mathcal{R}(r)$ can be expressed by (Ref. 11)

$$\mathcal{R}(r) = \frac{1 - \exp(-2\mu_v N)}{1 - \exp(-2\mu_v)} \int_0^1 [r(\tau) - r_a(\tau)]^2 d\tau \quad (51)$$

To minimize $\mathcal{R}(r)$, we must minimize

$$\int_0^1 [r(\tau) - r_a(\tau)]^2 d\tau$$

Thus, the minimization of $\mathcal{R}(r)$ becomes a simple problem, and the risk function can be obtained from Ref. 11. The procedure is to set the derivatives of $\mathcal{R}(r)$ with respect to a_j equal to zero, and the result can be obtained as follows (Ref. 11):

$$a = \alpha^{-1} d \quad (52)$$

where α is a matrix, whose elements are given by

$$\alpha_{kl} = \int_0^1 \exp(-2\mu_v |\tau|) \cos 2\pi k\tau \cos 2\pi l\tau d\tau \quad (53)$$

and d is a column vector, whose elements are given by

$$d_k = \int_0^1 r(\tau) \exp(-\mu_v |\tau|) \cos 2\pi k\tau d\tau \quad (54)$$

Furthermore, the following properties can easily be established:

$$\int_0^1 [r(\tau) - r_a(\tau)]^2 d\tau = \int_0^1 r^2(\tau) d\tau - \int_0^1 r_a^2(\tau) d\tau \quad (55)$$

$$\int_0^1 r^2(\tau) d\tau = \lim_{J \rightarrow \infty} \int_0^1 r_a^2(\tau) d\tau \quad (56)$$

III. Design of the Estimator

A. Design of a One-Step Predictor

Since we intend to utilize a digital computer for the estimation process, the model given by Eq. (26) is discretized, yielding

$$\begin{aligned} x(k+1) &= \bar{A}x(k) + \bar{B}u(k) \\ y(k) &= \bar{C}x(k) + v(k) \end{aligned} \quad (57)$$

In addition, the model given by Eq. (57) contains the observation noise element $v(k)$, which is assumed to be white, with mean zero and variance σ^2 . The parameters \bar{A} , \bar{B} , and \bar{C} are related to A , B , and C by (Ref. 10)

$$\begin{aligned}\bar{A} &= \exp\left(A \frac{T}{N}\right) \\ \bar{B} \bar{K} \bar{B}' &= \int_0^{T/N} \exp\left(A \frac{T}{N}\right) \exp(-As) BKB' \\ &\quad \times \exp(-A's) \exp\left(A' \frac{T}{N}\right) ds \\ \bar{C} &= C\end{aligned}\quad (58)$$

where K and \bar{K} are covariances of $u(t)$ and $u(k)$, respectively. The sampling interval utilized in the above discretization is chosen to be T/N . Thus, there will be N observations for each horizontal scan. Since there are N horizontal scan lines, the final discrete observation is on an $N \times N$ grid.

Example 4

Continuing example 3, we obtain

$$\begin{aligned}\bar{A} &= \begin{bmatrix} 0.996 & 0 & 0 & 0 & 0 \\ 0 & 0.983 & 0.031 & 0 & 0 \\ 0 & -1.22 & 0.97 & 0 & 0 \\ 0 & 0 & 0 & 0.926 & 0.03 \\ 0 & 0 & 0 & -4.77 & 0.913 \end{bmatrix} \\ \bar{B} \bar{K} \bar{B}' &= \begin{bmatrix} 0.02 & 0 & 0 & 0 \\ 0 & 0.02 & 0.12 & 0 \\ 0 & 0.12 & 0.60 & 0 \\ 0 & 0 & 0.01 & 0.07 \\ 0 & 0 & 0.07 & 0.49 \end{bmatrix} \\ \bar{C} = C &= [1 \quad 1 \quad 0 \quad 1 \quad 0]\end{aligned}$$

Utilizing the model given by Eq. (57) with parameters given by Eq. (58), a (one-step predictor) recursive estimator may be designed (Ref. 10). The equations are given for the sake of completeness.

$$\begin{aligned}\hat{x}(k+1) &= [\bar{A} - F(k)\bar{C}] \hat{x}(k) + F(k)y(k) \\ P(k+1) &= [\bar{A} - F(k)\bar{C}] P(k) [\bar{A} - F(k)\bar{C}]' \\ &\quad + \bar{B} \bar{K} \bar{B}' + F(k)F'(k)\sigma^2 \\ F(k) &= \bar{A}P(k)\bar{C}' [\bar{C}P(k)\bar{C}' + \sigma^2]^{-1}\end{aligned}\quad (59)$$

The (one-step predicted) estimate of the image is therefore

$$\bar{C}\hat{x}(k) \doteq \hat{y}(k)$$

that is, $\hat{y}(k)$ is the best estimate of $y(k)$, obtained recursively in real time, where $y(k)$ is the observation associated with the grid point immediately ahead of the scanner position.

Example 5

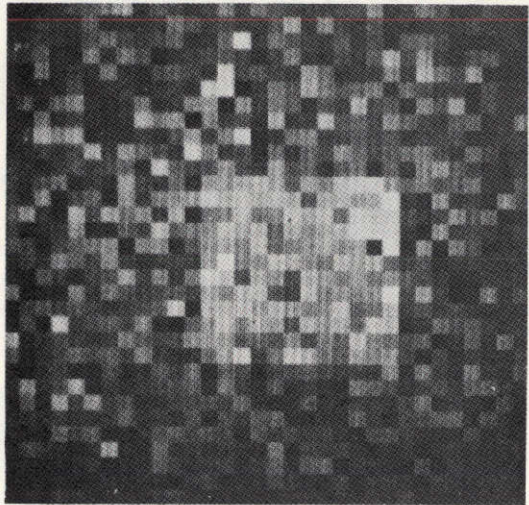
The signal $y(k)$ is generated by using the image described in the preceding example and adding white noise with variance σ^2 . Let us define a measure of signal-to-noise ratio by

$$\rho \doteq \frac{\text{peak-to-peak variation of signal}}{\sigma}$$

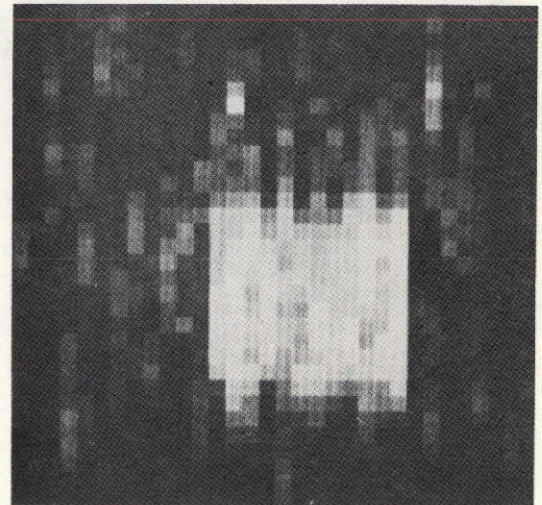
The peak-to-peak variation of the image is 7.1. Two values of ρ are considered here, namely 7.1/3 and 7.1/10; the corresponding values of $y(k)$ and their one-step predicted values $\hat{y}(k)$ are shown in Figs. 2a and b and 3a and b, respectively.

B. Implementation of Required Interpolation

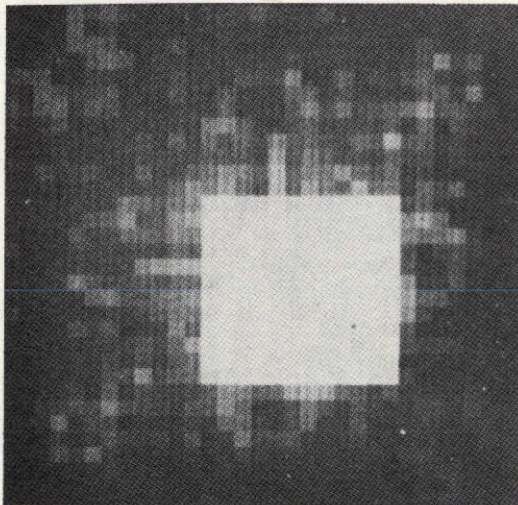
It is clear that image enhancement, from the point of view of scanner output, represents an interpolation problem; i.e., it is desired to determine the best estimate of $y(k)$, $0 \leq k \leq N$, given the observation $y(0), y(1), \dots, y(N)$. In general, the interpolation problem is far more complicated (Ref. 10) than standard Kalman filtering. However, since for the image enhancement considered here the length of the data is fixed (N) and, furthermore, the observation is usually available for additional repeated processing, it is possible to obtain two one-step predicted values of $y(k)$, denoted by $\hat{y}(k)$ and $\tilde{y}(k)$, one by running the scanner in one direction starting, for example, at the top left corner of the picture and the other by running the scanner in the reverse direction starting at bottom right corner. Associated with these estimates are estimation error variances denoted by $\hat{\sigma}^2(k) = \bar{C}\hat{P}(k)\bar{C}'$ and $\tilde{\sigma}^2(k) = \bar{C}\tilde{P}(k)\bar{C}'$, respectively. The two estimates must be combined to yield the optimal interpolated (smoothed)



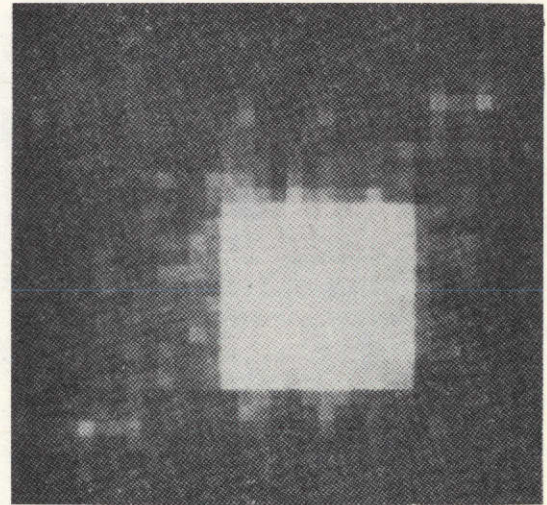
(a)



(b)



(c)



(d)

Fig. 2. Observation and estimates for $\rho = 7/3$

value $y^*(k)$. Thus, a brief discussion of combining two estimators is warranted.

Suppose we are given two state estimates, $\hat{x}(t)$ and $\tilde{x}(t)$, of the same state variable $x(t)$. There are two cases to consider; either $\hat{x}(t)$ and $\tilde{x}(t)$ are correlated or they are uncorrelated. We shall combine only the case in which both are uncorrelated; i.e.,

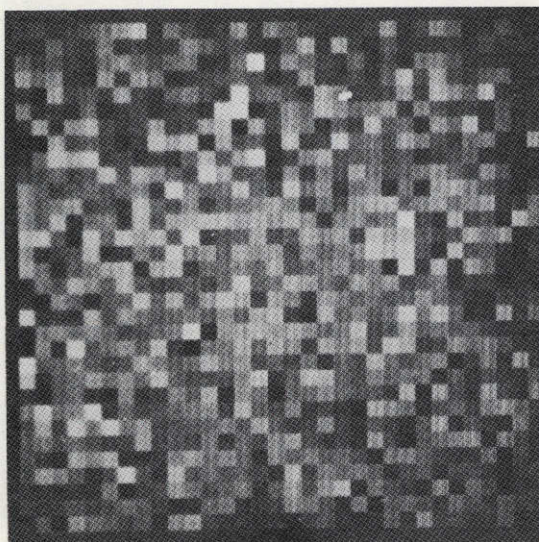
$$E[x - \hat{x}][x - \tilde{x}]' = 0 \quad (60)$$

In this case, the optimal estimate of x , denoted by $x^*(t)$, is given by

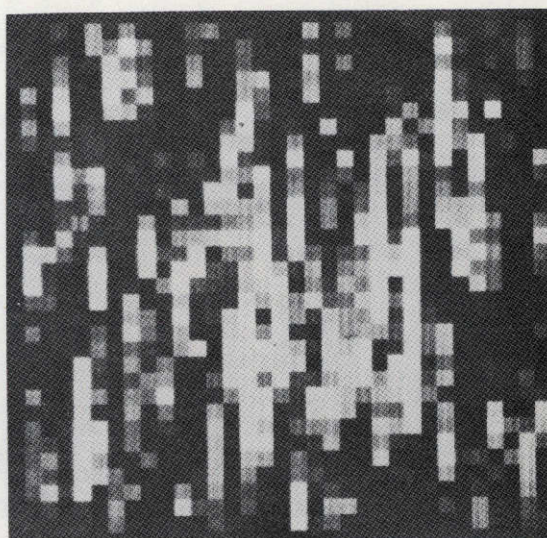
$$x^* = P^*(\hat{P}^{-1}\hat{x} + \tilde{P}^{-1}\tilde{x}) \quad (61)$$

$$P^* = (\hat{P}^{-1} + \tilde{P}^{-1})^{-1} \quad (62)$$

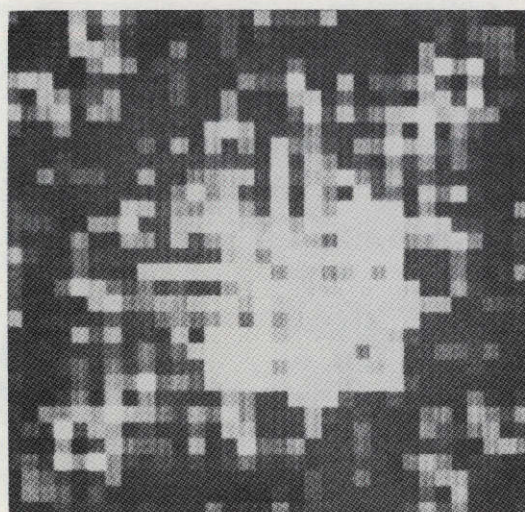
where \hat{P} and \tilde{P} are the error covariances of \hat{x} and \tilde{x} , respectively. Thus, applying Eqs. (60), (61), and (62) to obtain $\hat{y}(k) = \bar{C}\hat{x}$ and $\tilde{y} = \bar{C}\tilde{x}$ yields



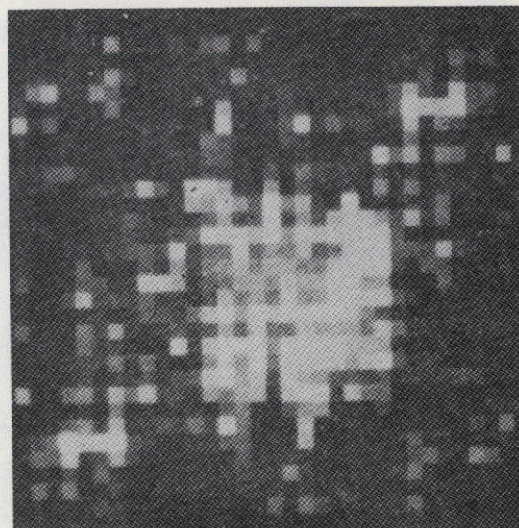
(a)



(b)



(c)



(d)

Fig. 3. Observation and estimates for $\rho = 7/10$

$$y^*(k) = \frac{\hat{\sigma}^2(k)}{\hat{\sigma}^2(k) + \bar{\sigma}^2(k)} \hat{y}(k) + \frac{\bar{\sigma}^2(k)}{\hat{\sigma}^2(k) + \bar{\sigma}^2(k)} \bar{y}(k) \quad (63)$$

Example 6

Considering the preceding example, the covariance $P(k)$ in Eq. (59) nearly reaches its steady-state value in about two or three scan lines. Consequently, $\hat{\sigma}(k) \approx \bar{\sigma}(k)$ for most of the picture, and Eq. (63) reduces to

$$y^*(k) \approx \frac{1}{2} [\hat{y}(k) + \bar{y}(k)] \quad (64)$$

Equation (64) was implemented, and the results for $\rho = 7.1/3$ and $7.1/10$ appear in Figs. 2c and 3c, respectively.

Careful observation of Figs. 2b and c (or 3b and c) reveals a consistent vertical correlation, which is attributed to the approximation of $r(\tau)$ by $r_a(\tau)$ (Fig. 1). The effect of this approximation is partially eliminated by

transposing the original picture and re-evaluating $y^*(k)$. The two values are then averaged and represent Figs. 2d and 3d for corresponding values of ρ .

IV. Conclusion

The feasibility of applying recursive (Kalman) filtering techniques to image processing has been established. Thus, the estimate at any one point does not require processing of all the data but only of the information stored by the point preceding it. The recursive procedure

is applicable to those images characterized statistically by means and correlation functions. It is important to note that the required computational time increases only linearly with the size of the picture (number of scan lines, number of discrete observations per line). A time-invariant dynamical model was chosen, leading to stationary statistics for the scanner output.

Improved performance is expected if the nonstationarity due to the scanner's periodic operation is considered. This improvement will be discussed in a later publication.

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